

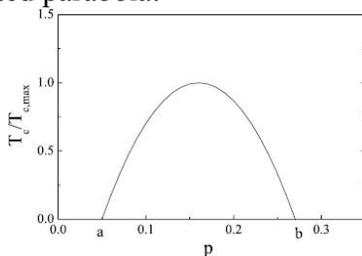
Physics 798I
Superconductivity
Fall 2012
Homework 4
Due October 26, 2012

1. Consider a model of a narrow-band superconductor in which $\hbar\omega_c$ is much larger than the width of the conduction band. You may neglect the differences in one-electron energies in the band ($\epsilon_k = \text{constant}$) and treat the problem as one with N electrons in a band of zero width containing $2M$ states (the two represents spins). Hint: Choose the chemical potential to be at the energy ϵ_k (this simplifies the expression for the kinetic energy!)

The point of this problem is to carry out the BCS calculation with a simple model that allows you to carry out all of the summations analytically, without ever having to do an integral on energy. Start by writing down the BCS ground state wavefunction, the pairing Hamiltonian, normalization constraint on the u 's and v 's, the total number of particles constraint, and the self-consistent gap equation, all in terms of the u 's and v 's. Note also that the BCS ground state solution does not have a fixed number of particles. Hence N is the only free parameter in this problem!

a) Find the BCS-like ground state (i.e. the u 's and v 's) for this problem, making the BCS approximation of a single value $-V$ for the coupling matrix element between all states. (Hint: In doing this, it is better to use BCS ideas, but not to try to take the limiting forms of the BCS results. Note that the level degeneracy implies that all states play an equal role.) If by chance you get the answer $u = v = 1/2^{1/2}$, use the values $u = (1 - N/2M)^{1/2}$ and $v = (N/2M)^{1/2}$ for part b. After finishing part b, you should be able to justify this substitution.

b) Find the gap parameter $\Delta(0)$ at $T=0$ in terms of M , N and V . For what values of N/M is it a maximum? What is that maximum? Sketch $\Delta(0)$ vs. N/M for values between 0 and 2. This has some resemblance to the T_c vs. doping (p) curve in the high- T_c cuprates, which looks like an inverted parabola:



c) Specializing to the case of a half-filled band, $N=M$, find the condensation energy at $T=0$.

d) For the case $N=M$, use the BCS gap equation to set up a simple transcendental equation determining $\Delta(T)$, and analytically determine T_c , the temperature at which $\Delta(T)$ goes to zero. Note that T_c is independent of $\hbar\omega_c$ in this model, so that there is no isotope effect. Find the ratio $\Delta(0)/kT_c$, and compare it with the BCS value of 1.76.

e) By graphical or numerical means, find $\Delta(t)/\Delta(0)$ for a number of values of $t=T/T_c$, and plot a rough sketch, for $N=M$. (This problem is mathematically equivalent to the determination of the molecular field solution for an $S=1/2$ ferromagnet.) By suitable expansions of the transcendental equation, find analytic limiting forms for $\Delta(t)/\Delta(0)$ valid near $t=0$ and $t=1$. Compare the high-temperature limit with the BCS result that $[\Delta(t)/\Delta(0)]^2 = 3.016 (1-t)$.

EXTRA CREDIT

Analytically carry out the definite integral on page 63, a few lines above Eq. (3.51), of Tinkham. Explicitly show your work.